1 a). i) $\frac{d}{dx}(\cos(kx)) = -k\sin kx \neq A\cos k$ => not an eigenfunction

ii) $\frac{d^2}{dx^2}\left(\exp(-ax^2)\right) = \frac{d}{dx}\left(\frac{d(-ax^2)}{dx}e^{-ax^2}\right) = \frac{d}{dx}\left(-2axe^{-ax^2}\right)$

$= -2ax(-2axe^{-ax^2}) + (-2ae^{-ax^2}) = (4a^2x^2 - 2a)e^{-ax^2}$

$\neq Ae^{-ax^2}$

=> not an eigenfunction

iii) $\frac{d^2}{dx^2}(\cos(kx)) = \frac{d}{dx}(-k\sin(kx)) = -k^2\cos kx = A\cos kx$

$A = -k^2$, eigenfunction

iv) $\frac{d^2}{dx^2}\sin kx = -k^2\sin kx$

$A = -k^2$, eigenfunction

v) $\hat{i}(x^3 - kx) = (-x)^3 - k(-x) = -x^3 + kx = (-1)(x^3 - kx)$

$A = -1$, eigenfunction

b) $\frac{d^2}{dx^2}\left(\cos kx + \sin kx\right) = \frac{d^2}{dx^2}\cos kx + \frac{d^2}{dx^2}\sin kx = -k^2\cos kx + -k^2\sin kx$

$= -k^2(\cos kx + \sin kx)$ $A = -k^2$, eigenfunction

$\frac{d^2}{dx^2}\left(\cos kx + \exp(-ax^2)\right) = -k^2\cos kx + (4a^2x^2 - 2a)e^{-ax^2}$

$\neq Af(x)$; not an eigenfunctions

Sums of eigenfunctions are eigenfunctions

2) a) $\psi_1(x) = \frac{2x}{a} e^{-x^2/2a^2}$

from integral table

$$\int_{-\infty}^{\infty} \psi_1^* \psi_1 \, dx = \frac{4}{a^2} \int_{-\infty}^{\infty} x^2 e^{-x^2/a^2} = \frac{4}{a^2} \cdot 2 \cdot \frac{1}{4} \sqrt{\frac{\pi}{(1/a^2)^3}} = \frac{2}{a^2} a^3 \sqrt{\pi} = 2a\sqrt{\pi}$$

$$\Psi_1(x) = \left(\frac{1}{2a\sqrt{\pi}}\right)^{1/2} \frac{2x}{a} e^{-x^2/2a^2}$$

$$\psi_2(x) = \left(4\left(\frac{x}{a}\right)^2 - 2\right) e^{-x^2/2a^2}$$

$$\int_{-\infty}^{\infty} \left(4\left(\frac{x}{a}\right)^2 - 2\right) e^{-x^2/2a^2} \cdot \left(4\left(\frac{x}{a}\right)^2 - 2\right) e^{-x^2/2a^2} \, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{16x^4}{a^4} - \frac{16x^2}{a^2} + 4\right) e^{-x^2/a^2} \, dx$$

$$= \int_{-\infty}^{\infty} \left(\frac{16x^4}{a^4} e^{-x^2/a^2} - \frac{16x^2}{a^2} e^{-x^2/a^2} + 4e^{-x^2/a^2}\right) dx$$

$$= \frac{2 \cdot 16}{a^4} \cdot \frac{(2 \cdot 2)!}{2 \cdot 2^5} \sqrt{\frac{\pi}{\left(\frac{1}{a^2}\right)^5}} - \frac{2 \cdot 16}{a^2} \frac{1}{4} \sqrt{\frac{\pi}{(1/a^2)^3}} + 2 \cdot 4 \sqrt{\frac{\pi}{1/a^2}}$$

$$= 12a\sqrt{\pi} - 8a\sqrt{\pi} + 8a\sqrt{\pi} = 12a\sqrt{\pi}$$

$$\Psi_2(x) = \left(\frac{1}{12a\sqrt{\pi}}\right)^{1/2} \left(4\left(\frac{x}{a}\right)^2 - 2\right) e^{-x^2/2a^2}$$

b $\int_{-\infty}^{\infty} \psi_1 \psi_2 \, dx = \int_{-\infty}^{\infty} \frac{2x}{a} e^{-x^2/2a^2} \left(4\left(\frac{x}{a}\right)^2 - 2\right) e^{-x^2/2a^2} dx$

$= \int_{-\infty}^{\infty} \underbrace{\frac{8x^3}{a^3}}_{\text{odd}} \underbrace{e^{-x^2/a^2}}_{\text{even}} dx - \int_{-\infty}^{\infty} \underbrace{\frac{4x}{a}}_{\text{odd}} \underbrace{e^{-x^2/a^2}}_{\text{even}} dx = 0$

3) a) $\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi_1(x) \, x^2 \, \Psi_1(x) \, dx$

$\frac{1}{2a\sqrt{\pi}} \cdot \frac{4}{a^2} \int_{-\infty}^{\infty} x e^{-x^2/2a^2} x^2 x e^{-x^2/2a^2} dx = \frac{2}{a^3\sqrt{\pi}} \int_{-\infty}^{\infty} x^4 e^{-x^2/a^2} dx$

$= \frac{2}{a^3\sqrt{\pi}} \frac{2 \cdot (2 \cdot 2)!}{2! \, 2^5} \sqrt{\frac{\pi}{\left(\frac{1}{a^2}\right)^5}} = \frac{3}{2} a^2$

b) $\langle p \rangle = \int_{-\infty}^{\infty} \Psi_1(x) \, \hat{p} \, \Psi_1(x) \, dx = \frac{-i\hbar}{2a\sqrt{\pi}} \frac{4}{a^2} \int_{-\infty}^{\infty} x e^{-x^2/2a^2} \frac{d}{dx} x e^{-x^2/2a^2} dx$

$= \frac{4i\hbar}{2a^3} \int_{-\infty}^{\infty} x e^{-x^2/2a^2} \left(e^{-x^2/2a^2} + x(-2x) e^{-x^2/2a^2} \right) dx$

$= \frac{-2i\hbar}{a^3} \left[\int_{-\infty}^{\infty} \underbrace{x}_{\text{odd}} \underbrace{e^{-x^2/a^2}}_{\text{even}} dx - 2\int_{-\infty}^{\infty} \underbrace{x^3}_{\text{odd}} \underbrace{e^{-x^2/a^2}}_{\text{even}} dx \right] = 0$

Mathcad knows Hermite polynomials. Here is the normalized version of ours:

$$\psi_1(x,\alpha) := \left(\frac{1}{4\pi}\right)^{\frac{1}{4}} \cdot \alpha^{\frac{-3}{2}} \cdot \mathrm{Her}(1,x) \cdot e^{\frac{-x^2}{2\alpha^2}}$$

Verified here:

$$g(\alpha) := \int_{-\infty}^{\infty} \psi_1(x,\alpha) \cdot \psi_1(x,\alpha)\,dx \qquad g(1) = 1 \qquad g(2) = 1 \qquad g(0.1) = 1 \qquad g(99) = 1$$

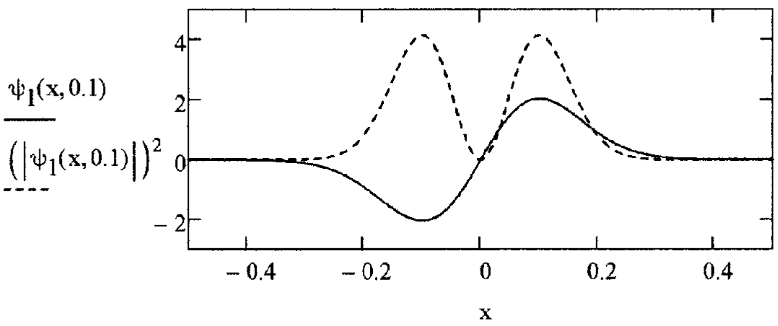
3

$$\int_{-\infty}^{\infty} \psi_1(x,0.234) \cdot x^2 \cdot \psi_1(x,0.234)\,dx = 0.082 \qquad \frac{0.082}{0.234^2} = 1.498$$

$$\int_{-\infty}^{\infty} \psi_1(x,0.1) \cdot \left(\frac{d}{dx}\psi_1(x,0.1)\right) dx = 0$$

4 If alpha is in meters, x will be in meters

a, b



c

i $$\int_{0}^{\infty} \psi_1(x,0.1) \cdot \psi_1(x,0.1)\,dx = 0.5$$

ii $$\int_{-\infty}^{0} \psi_1(x,0.1) \cdot \psi_1(x,0.1)\,dx = 0.5$$

We already knew it was going to be split 50/50 about the y axis.

iii $$\int_{-0.01}^{0.01} \psi_1(x,0.1) \cdot \psi_1(x,0.1)\,dx = 7.478 \times 10^{-4}$$

There is a node at the origin, so there is a low probability of finding it there.

iv $$\int_{0.09}^{0.11} \psi_1(x,0.1) \cdot \psi_1(x,0.1)\,dx = 0.082$$