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Course 565/665 Lecture Number _____ Date 2/11/03

Lecturer Dr. Silvia Cavagnero Note Taker Eric Fulmer

Partial Derivatives

$\Delta f = f(x) - f(a) = \ldots$ [from yesterday.]

But in the case of a function with 2 independent variables,

$$\Delta f = f(x,y) - f(a,b) = \left(\frac{\partial f}{\partial x}\right)_y \Delta x + \left(\frac{\partial f}{\partial y}\right)_x \Delta y$$

$$+ \frac{1}{2!}\left[\left(\frac{\partial^2 f}{\partial x^2}\right)_y \Delta x^2 + \left(\frac{\partial^2 f}{\partial y^2}\right)_x \Delta y^2 + 2\left(\frac{\partial^2 f}{\partial x \partial y}\right)\Delta x \Delta y\right] + \ldots$$

The Δf represents a finite increment.

For $\left.\begin{matrix} a \to x \\ b \to y \end{matrix}\right\}$ $\Delta f \to df$, where df represents an infinitesimally small (very, very small) amount.

Therefore, $dy^2, dx^2, dxdy, \ldots \ll dx, dy$

and we can neglect the 2nd order and higher order terms.

Thus, $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$

The above equation is a <u>Total Differential</u>.

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For a function depending on "t" independent variables,

$$df = \sum_{i=1}^{t} \left(\frac{\partial f}{\partial x_i}\right)_{x_j \neq x_i} dx_i \qquad \text{(Total Differential)}$$

Multivariate Functions

Critical Points

- Maxima
- Minima
- Saddle Points

For a function that depends on only 1 independent variable, the first derivative is equal to zero at a critical point.

$$\left(\frac{df}{dx}\right)_{x^*} = 0 \quad \text{at a critical point.}$$

$\left(\frac{d^2f}{dx^2}\right) < 0 \Rightarrow$ ~~probably~~ a maximum at x^*

$\left(\frac{d^2f}{dx^2}\right) > 0 \Rightarrow$ ~~probably~~ a minimum at x^*

$$df = \left(\frac{df}{dx}\right)_y dx + \left(\frac{df}{dy}\right)_x dy = 0 \quad \text{at } x^*, y^*$$

Since dx and dy cannot be equal to zero, the first derivatives must be equal to zero.

Thus, for a critical point for $f(x,y)$:

$$\boxed{\left(\frac{\partial f}{\partial x}\right)_y = 0 \quad \underline{\text{AND}} \quad \left(\frac{\partial f}{\partial y}\right)_x = 0}$$

If

$\left(\frac{\partial^2 f}{\partial x^2}\right)_y , \left(\frac{\partial^2 f}{\partial y^2}\right)_x > 0$, probably a minimum.

$\left(\frac{\partial^2 f}{\partial x^2}\right)_y , \left(\frac{\partial^2 f}{\partial y^2}\right)_x < 0$, probably a minimum.

We MUST check the <u>HESSIAN</u>, as well:

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{y^*,x^*} \left(\frac{\partial^2 f}{\partial y^2}\right)_{x^*,y^*} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 \quad \begin{cases} > 0, & \text{an extremum} \\ \leq 0, & \text{not an extremum.} \end{cases}$$

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Example $f(x,y) = z = x^2 + y^2 - 4xy$

Critical Point?

$$\left(\frac{\partial f}{\partial x}\right)_y = 2x - 4y = 0$$

$$\left(\frac{\partial f}{\partial y}\right)_x = 2y - 4x = 0$$

Solving these equations, the only values that allow these equations to be equal to zero is for x and y to equal zero.

$$2x - 4y = 0$$
$$2x = 4y$$
$$x = 2y$$
$$x = 2(0)$$
$$x = 0$$

$$2y - 4x = 0$$
$$2y - 4(2y) = 0$$
$$-6y = 0$$
$$y = 0$$

The critical point is at $f(0,0)$.

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$$\left(\frac{\partial^2 f}{\partial x^2}\right)_y = \left(\frac{\partial^2 f}{\partial y^2}\right)_x = 2$$ $\Rightarrow$ Either a saddlepoint or a minimum.

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial}{\partial y}\left(2x - 4y\right)$$
$$= -4$$

<u>HESSIAN</u>: $H = (2)(2) - (-4)^2$
$H = 4 - 16 = -12$

Thus, the HESSIAN is negative. This point $f(0,0)$ must therefore be a <u>SADDLEPOINT</u> and not an extremum.

We Shall Skip LaGrange Multipliers

<u>Multivariate Functions with constraints</u>

- ~~Add~~ a constraint equation $g(x,y)$. This equations gives a definite relationship between the two independent variables. These variables are no longer independent when there is a constraint.

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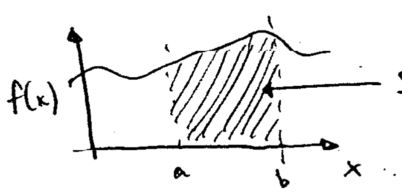
To Get a Critical Point with a Constraint

① $df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy = 0$

② $dg = \left(\frac{\partial g}{\partial x}\right)_y dx + \left(\frac{\partial g}{\partial y}\right)_x dy = 0$

Example 5.5 ⇒ Do At Home

Integration - Finding the Area below a given curve



$$\Omega = \int_a^b f(x)\, dx$$

For a Multivariate Function, $f(x,y)$

$$\Delta f = \int_A^B df = \int_{x_A}^{x_B}\int_{y_A}^{y_B} dx\,dy$$

Must Worry about Paths with Multivariate Integration