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Course 565/665 Lecture Number ______ Date 2/13/03

Lecturer Dr. Silvia Caragnero Note Taker Eric Fulmer

LAST TIME

Integration of multivariate functions

- Pathways may or may not make a difference.

Pathway Dependent Functions $\longrightarrow$ NONSTATE FUNCTIONS

Pathway Independent Functions $\longrightarrow$ STATE FUNCTIONS

Given $f(x,y)$

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$$\Delta f = \iint_A^B df = \int_{x_A}^{x_B} dx \int_{y_A}^{y_B} dy$$

If the above $\Delta f = f(x_B, y_B) - f(x_A, y_A)$, then $f(x,y)$ is a state function.

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Properties of State Functions

- If f is a state function, df is known as an exact differential.
- $\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial^2 f}{\partial y \partial x}\right)$

If we define $s(x,y) = \left(\frac{\partial f}{\partial x}\right)_y$ and $t(x,y) = \left(\frac{\partial f}{\partial y}\right)_x$, then Euler's Theorem says that the following is true if $f(x,y)$ is a state function.

$$\left(\frac{\partial s}{\partial y}\right)_x = \left(\frac{\partial^2 f}{\partial y \partial x}\right) = \left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial t}{\partial x}\right)_y$$

Example

$$f(x,y) = xy$$

$$df = y\,dx + x\,dy = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

Then, $\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \left(\frac{\partial (y)}{\partial y}\right)_x = 1$

$\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial (x)}{\partial x}\right)_y = 1$

These are therefore State Functions.

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Example: Is the following an exact differential?

$$df = 6xy^3\,dx + 9x^2y^2\,dy$$

$$= \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$$\left(\frac{\partial^2 f}{\partial y \partial x}\right) = \left(\frac{\partial (6xy^3)}{\partial y}\right)_x = 18xy^2$$

$$\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \left(\frac{\partial (9x^2y^2)}{\partial x}\right)_y = 18xy^2$$

$$f(x,y) = 3x^2y^3 + \text{constant}$$

This function is therefore a State Function.

Chain Rule

Given $f(x,y)$ and $x = x(u)$, $y = y(u)$

We can write: $dx = \frac{dx}{du}\,du$, $dy = \frac{dy}{du}\,du$

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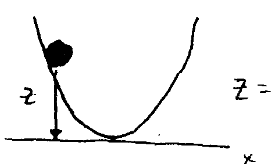
$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

$$= \left[\left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{du} + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{du}\right] du$$

$$\frac{df}{du} = \left(\frac{\partial f}{\partial x}\right)_y \frac{dx}{du} + \left(\frac{\partial f}{\partial y}\right)_x \frac{dy}{du}$$

Example

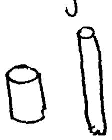


$z = x^2$

$$U = mgz$$
$$z = x^2$$

$$\frac{dU}{dx} = \left(\frac{\partial U}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = (mg)(2x) = 2mgx$$

Rearranging Differentials: The constant volume of a given cylinder. We want to change r and h but not change the total value of the function (the volume V).



$V = \pi r^2 h$

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Given $V = \pi r^2 h$ and we want $dV = 0$,

$$dV = \left(\frac{\partial V}{\partial r}\right)_h dr + \left(\frac{\partial V}{\partial h}\right)_r dh = 0$$

$$\left(\frac{\partial V}{\partial r}\right)_h dr = -\left(\frac{\partial V}{\partial h}\right)_r dh$$

$$\frac{dr}{dh} = -\frac{(\partial V/\partial h)_r}{(\partial V/\partial r)_h} = -\left(\frac{\partial V}{\partial h}\right)_r \left(\frac{\partial r}{\partial V}\right)_h$$

This looks strikingly similar to our Chain Rule.

$$\frac{dr}{dh} = \frac{-\pi r^2}{2\pi r h} = \frac{-r}{2h}$$

If the Volume stays constant, and we double the length of r, what happens to h?

$$\frac{dh}{dr} = \frac{-2h}{r} \qquad \text{Separate Variables}$$

$$\int_{r_1}^{r_2} \frac{dr}{r} = -\int_{h_1}^{h_2} \frac{dh}{2h}$$

$$\ln\left(\frac{r_2}{r_1}\right) = -\frac{1}{2}\ln\left(\frac{h_2}{h_1}\right) \qquad \frac{h_2}{h_1} = \left(\frac{r_1}{r_2}\right)^2 = \frac{1}{4}$$