Shot noise

Photodiodes (PD), photomultiplier tubes (PMT) and charge coupled devices (CCD) all work on the basic principle that when photons strike surfaces they generate free electrons and the movement of the free electrons results in a measurable current. Shot noise in the current is due to the statistical nature of the generation of the free electrons. In any given time interval, there will be random fluctuations in the number of free electrons generated as the photons strike that surface. These fluctuations will follow Poisson statistics, which tells us that the uncertainty in the number of events, n, that occur in a given time interval is given simply the square root of n, or $\sigma_n = \sqrt{n}$. For example, if on average 100 electrons are generated in a given time interval, the standard deviation of a series of counts over that time interval would be ±10 electrons.

In addition to the current generated from the photons of the light source, there is also a contribution to the current due to the thermionic emission of electrons from the cathode material. This is called the dark current. To properly consider contributions to the shot noise from both sources, again using Poisson statistics, you would add the variances of the two noise sources to get

$$\sigma_{total}^2 = \sigma_{dark}^2 + \sigma_{light}^2 = n_{dark} + n_{light} \quad \text{or} \quad \sigma_{total} = \sqrt{n_{dark} + n_{light}} \,.$$

We can simplify the analysis by looking at two limiting cases. In the high light limit, as in an absorption experiment, $n_{light} >> n_{dark}$ and we only need to consider the photon contributions to the shot noise. In the low light limit, as in an emission experiment (assuming zero background light), $n_{light} << n_{dark}$ and we only need to consider the dark current contribution to the shot noise.

During a measurement time of Δt, the number of events (creation of free electrons) , is given by:

$n = \frac{i \ \Delta t}{e}$ where i is the measured current and e is the charge of an electron (e = 1.6x10^{-19}C)

Shot noise is based on Poisson Statistics so n measurements would have a standard deviation of

$$\sqrt{n} = \sqrt{\frac{i\Delta t}{e}} \,.$$

The standard deviation of the current is

$$i_{noise} = \frac{\sqrt{n}\ e}{\Delta t} = \sqrt{\frac{i\ \Delta t}{e}} \cdot \frac{e}{\Delta t} = \sqrt{\frac{i\ e}{\Delta t}} = \underline{\underline{\sqrt{2ie\Delta f}}}$$

If the bandwidth of the measurement, Δf, is given by $\Delta f = \frac{1}{2\Delta t}$ (see below).

For a photomultiplier tube, the current is measured at the anode, but the anode current is the result of the multiplication of electrons emitted at the cathode. The number of electrons emitted at the cathode is $n = \frac{i_{anode}\Delta t}{G \cdot e}$ where G is the overall gain of the PMT. The standard deviation is then $\sqrt{n} = \sqrt{\frac{i_{anode}\Delta t}{e}}$ and the fluctuation in the anode current is given by

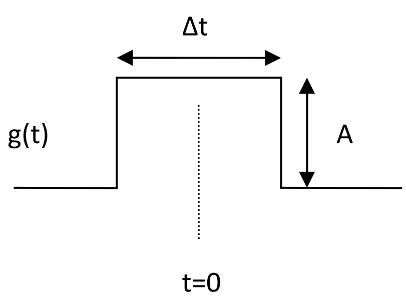
$$i_{noise} = \frac{\sqrt{n}\ G \cdot e}{\Delta t} = \sqrt{\frac{i_{anode}\Delta t}{G \cdot e}} \cdot \frac{G \cdot e}{\Delta t} = \sqrt{\frac{i_{anode} G \cdot e}{\Delta t}} = \underline{\underline{\sqrt{2 i_{anode} G \cdot e \cdot \Delta f}}}$$

For a PMT, there are also contributions to the shot noise due to the multiplicative effect of the secondary electrons emitted at each dynode. This can be shown to contribute a term in the noise equation of $\sqrt{\frac{\delta}{\delta - 1}}$ where δ is the secondary emission yield. As long as δ is much larger than one, the dynode contribution to the noise is negligible. [i]

But where does the factor of two come from in the relationship between Δt and Δf?

A measurement for a time of Δt is described in the time domain as:

$$g(t) = A \text{ from } t = -\frac{\Delta t}{2} \text{ to } \frac{\Delta t}{2} \text{ and 0 everwhere else}$$



Here, we define t= 0 at the center of the interval, to make the Fourier transform easier. In an even function only cosine terms are needed.

Taking the Fourier transform will provide the frequency distribution expected from a measurement done over Δt-

$$G(f) = \int_{-\infty}^{\infty} g(t)\cos(2\pi \cdot f \cdot t)dt \;=\; \int_{-\Delta t/2}^{\Delta t/2} A\cos(2\pi \cdot f \cdot t)dt \;=\; \left[\frac{A}{2\pi \cdot f}\sin\left(2\pi \cdot f \cdot \frac{\Delta t}{2}\right) - \frac{A}{2\pi \cdot f}\sin\left(2\pi \cdot f \cdot \frac{-\Delta t}{2}\right)\right]$$

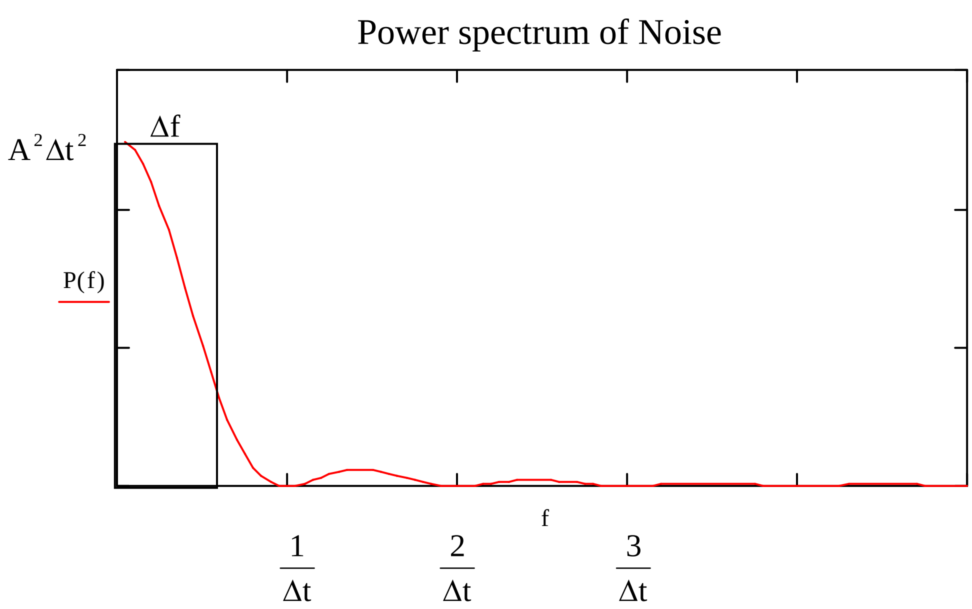
or

$$G(f) = \frac{A}{\pi f}\sin(\pi \cdot f \cdot \Delta t)$$

The expression can also be written as $G(f) = (A \cdot \Delta t)\dfrac{\sin(\pi \cdot f \cdot \Delta t)}{(\pi \cdot f \cdot \Delta t)}$ which you may recognize as a sinx/x or sinc function. Square to get the Power spectrum

$$P(f) = G(f)^2 = (A \cdot \Delta t)^2 \frac{\sin^2(\pi \cdot f \cdot \Delta t)}{(\pi \cdot f \cdot \Delta t)^2}$$

And the power spectrum plot looks like



Define the noise equivalent bandwidth, Δf, as the width of a rectangle band of frequencies whose amplitudes are equivalent to the most intense frequency and whose total area is equivalent to the actual power spectrum.

Total area of P(f)

$$\text{Area} = \int_0^\infty (A \cdot \Delta t)^2 \frac{\sin^2(\pi \cdot f \cdot \Delta t)}{(\pi \cdot f \cdot \Delta t)^2} df = \frac{A^2}{\pi^2} \int_0^\infty \frac{1}{f^2} \sin^2(\pi \cdot f\Delta t) df^* = \frac{A^2 \Delta t}{2}$$

Draw a rectangle of area $A^2 \Delta t^2 \cdot \Delta f$ and set it equal to total power area

$$A^2 \Delta t^2 \cdot \Delta f = \frac{A^2 \Delta t}{2}$$ and solve to get

$$\Delta f = \frac{1}{2\Delta t}$$ (Whew!!)

*The evaluation of the definite integral From the CRC handbook $\int_o^\infty \frac{1}{x^2} \sin^2(px)dx = \frac{\pi}{2} p$

What if you do not know Δt, but you know a single RC time constant or the 3db cutoff frequency associated with your detection electronics.

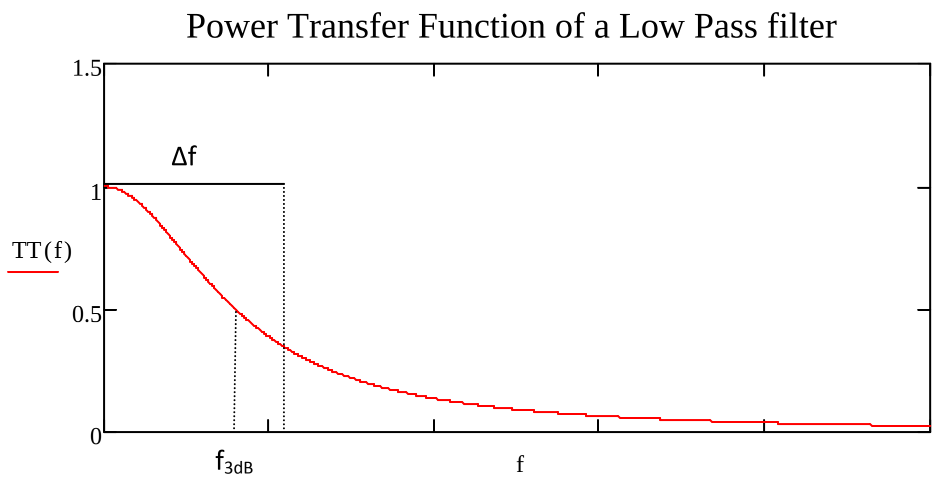
The transfer function for a low pass filter with a single R and C is given by:

$$T = \frac{V_{out}}{V_{in}} = \frac{1}{\sqrt{1 + f^2 (2\pi RC)^2}}$$

In terms of Power, the transfer function is written as:

$$\frac{P_{out}}{P_{in}} = \left(\frac{V_{out}}{V_{in}} \right)^2 = TT = \frac{1}{1 + f^2 (2\pi RC)^2} = \frac{1}{1 + \left(\frac{f}{f_{3db}} \right)^2}$$ where $f_{3db} = \frac{1}{2\pi RC}$

Graphically, the power transfer looks like



Again we can define the noise equivalent bandwidth, Δf, as the width of a rectangle band of frequencies that contain an equivalent power as throughput power of the low pass filter. So that

$$\Delta f \cdot 1 = \int_0^{\infty} \frac{1}{1+\left(\frac{f}{f_{3db}}\right)^2} df = f_{3db} \tan^{-1}\left(\frac{f}{f_{3db}}\right)\Big|_0^{\infty}$$

$$\Delta f = f_{3db}\frac{\pi}{2} = 1.57 \cdot f_{3db}$$

Or in terms of RC, $\Delta f = f_{3db}\frac{\pi}{2} = \left(\frac{1}{2\pi RC}\right)\frac{\pi}{2} = \frac{1}{4RC}$

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[i] John Wright, Notes for Chem 621.